University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO EXAMINATION 3

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for three $8\frac{1}{2}\times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (50 points)
A satellite is in elliptical orbit about the earth (neglect any effects of the moon or sun). Its radius r is proportional to

$$r \propto \frac{1}{1+\epsilon\cos\theta}\ ,$$

where θ is the azimuthal angle of the orbit, and ϵ is the ellipse's *eccentricity*. For simplicity take $r_{\text{max}} = 3r_{\text{min}}$, so that $\epsilon = \frac{1}{2}$.
(**a**) (10 points)
Using any relevant theorem(s), write down the ratio $-\langle T\rangle/\langle U\rangle$, where T and U are the satellite's kinetic and potential energies, and $\langle\rangle$ is the time average over one full orbit.
Solution:
According to the Virial Theorem, if the (attractive) force varies as r^{-n},

$$\langle T\rangle = -\frac{n-1}{2}\langle U\rangle = -\tfrac{1}{2}\langle U\rangle\ .$$

(The above earns full credit. If you're lacking the details of the Virial Theorem, you need only recall that $-\langle T\rangle/\langle U\rangle$ depends only on the exponent of r in the force law. Consider a circular orbit in a $-k/r^2$ force field:

$$\begin{aligned}\frac{mv^2}{r} &= \frac{k}{r^2}\\ \tfrac{1}{2}mv^2 &= \tfrac{1}{2}\frac{k}{r}\\ T &= -\tfrac{1}{2}U\ .\end{aligned}$$

On average, this is true also for an elliptical orbit, since the force law is the same.)

(**b**) (20 points)
When $r = r_{\text{max}}$, what is $-T/U$? [*Hint*: the satellite's total energy is inversely proportional to the semimajor axis of its orbit. If you don't remember the constant of proportionality, you can deduce it by considering the special case of a circular orbit.]
Solution:
Take a to be the ellipse's semimajor axis. It is related to r_{max} by

$$\begin{aligned}2a &= r_{\text{min}} + r_{\text{max}}\\ &= r_{\text{max}}\left(\tfrac{1}{3}+1\right)\\ a &= \tfrac{2}{3}r_{\text{max}}\ .\end{aligned}$$

If the (attractive) force is k/r^2, the satellite's total energy is

$$E = -\frac{k}{2a}\ .$$

(Lacking the constant of proportionality $-k/2$, you may deduce it from the circular orbit considered in the solution to (**a**):

$$\begin{aligned}E &= T+U\\ &= -\tfrac{1}{2}U + U\\ &= \tfrac{1}{2}U\\ &= -\frac{k}{2r}\ ,\end{aligned}$$

when a reduces to r in that special case.) Solving for the kinetic energy at $r = r_{\max}$,

$$\begin{aligned} T &= E - U \\ &= -\frac{k}{2a} - \frac{-k}{r_{\max}} \\ &= -\frac{k}{2a} - \frac{-k}{\frac{3}{2}a} \\ &= \frac{k}{6a} \\ \frac{T}{U} &= \frac{k/6a}{-2k/3a} \\ T &= -\tfrac{1}{4}U \ . \end{aligned}$$

(**c**) (20 points)
When $r = r_{\max}$, a rocket on board the satellite fires a very brief burst, consuming fuel of negligible mass. Immediately after the burst, the satellite's total energy (normalized to zero at $r = \infty$) changes by a factor C, but its direction of motion remains the same; the satellite's orbit becomes *circular*. Solve for C.
Solution:
Immediately after the rocket fires, the satellite is still at the same radius (otherwise it would undergo infinite acceleration in the limit that the burst duration vanishes). Therefore, since it is now in circular orbit,

$$E' = -\frac{k}{2r_{\max}} \ .$$

The original total energy was

$$E = -\frac{k}{2a} \ .$$

Their ratio is

$$C = \frac{E'}{E} = \frac{a}{r_{\max}} = \tfrac{2}{3} \ .$$

Note that the total energy is reduced *in magnitude* by the rocket burst. However the gain in total energy is *positive* because the total energy remains negative in sign.

2. (50 points)
When undriven, an undamped oscillator (*i.e.* a mass on a spring) satisfies the equation

$$\ddot{x} + \omega_0^2 x = 0 \ ,$$

where ω_0 is a positive constant. For $t < 0$ it is at rest at the origin: $x(t < 0) = 0$.
(**a**) (20 points)
For this part, suppose that the mass is given a *quick tap* at $t = 0$, *i.e.*

$$\begin{aligned} x(t = 0^+) &= 0 \\ \dot{x}(t = 0^+) &= v_0 \ , \end{aligned}$$

where v_0 is a positive constant. Solve for $x(t)$ for $t > 0$. [*Hint*: your solution should be equivalent to $v_0\, G(t)$, where $G(t)$ is the *Green function* for this oscillator.]
Solution:
The general solution is

$$x_h = B \cos(\omega_0 t + \beta) \ .$$

Applying the initial conditions at $t = 0$,

$$\begin{aligned} 0 &= x_h(0) \\ &= B \cos\beta \\ \Rightarrow \beta &= \tfrac{\pi}{2} \\ \Rightarrow x_h(t) &= -B \sin\omega_0 t \\ v_0 &= \dot{x}_h(0) \\ &= -B\omega_0 \\ \Rightarrow B &= -\frac{v_0}{\omega_0} \\ \Rightarrow x_h(t) &= \frac{v_0}{\omega_0} \sin\omega_0 t \ . \end{aligned}$$

Alternatively, taking advantage of the hint, you may obtain the same result as the limit of the Green function for the underdamped oscillator as $\gamma \to 0$.
(**b**) (30 points)
For this part, suppose instead that the mass is given a *steady push* that begins at $t = 0$ and lasts for one period. That is, suppose that the force F on the mass, divided by the mass m, is such that $F/m = a(t)$, where

$$\begin{aligned} a(t) &= a_0 \quad (0 < t < \tfrac{2\pi}{\omega_0}) \\ &= 0 \qquad \text{otherwise} \ , \end{aligned}$$

where a_0 is a positive constant. Solve for $x(t)$ after the push is finished, *i.e.* for $t > 2\pi/\omega_0$.

Solution:

Method 1. Simple argument.

During the push, irrespective of details that depend on initial conditions, the motion must be periodic with period $2\pi/\omega_0$. Therefore, after one period, the system must revert to its initial conditions $x = 0$, $\dot{x} = 0$. Given these conditions at $t = 2\pi/\omega_0$, after the driving force has vanished the mass must remain in the same conditions, *i.e.* at rest at the equilibrium position $x = 0$.

Method 2. Green function.

Using the solution $x_a(t)$ and the hint from (**a**), the Green function for this oscillator is

$$\begin{aligned} G(t) &= \frac{x_a(t)}{v_0} \\ &= \frac{\sin\omega_0 t}{\omega_0} \quad (t > 0) \\ &= 0 \quad \text{otherwise} \\ G(t-t') &= \frac{\sin\omega_0(t-t')}{\omega_0} \quad (t > t') \\ &= 0 \quad \text{otherwise} . \end{aligned}$$

The Green function yields an integral equation for $x(t)$:

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} dt'\, a(t')\, G(t-t') \\ &= \int_{-\infty}^{t} dt'\, a(t')\, \frac{\sin\omega_0(t-t')}{\omega_0} \\ x\big(t > \tfrac{2\pi}{\omega_0}\big) &= \int_{0}^{2\pi/\omega_0} dt'\, a_0\, \frac{\sin\omega_0(t-t')}{\omega_0} . \end{aligned}$$

For any value of $t > \frac{2\pi}{\omega_0}$, this is proportional to the integral of a sinusoidal function over one period, which must vanish. Therefore

$$x\big(t > \tfrac{2\pi}{\omega_0}\big) = 0 .$$

Method 3. Brute force solution of equations of motion.

During the push, the equation of motion is

$$\ddot{x} + \omega_0^2 x = a_0 .$$

The general solution is the sum of x_h and x_p, where

$$\begin{aligned} x_h &= B\cos(\omega_0 t + \beta) \\ x_p &= \frac{a_0}{\omega_0^2} . \end{aligned}$$

Applying the initial conditions at $t = 0$, ,

$$\begin{aligned} 0 &= \dot{x}_h(0) + \dot{x}_p(0) \\ &= -B\omega_0 \sin\beta \\ \Rightarrow \beta &= 0 \\ 0 &= x_h(0) + x_p(0) \\ &= B\cos\beta + \frac{a_0}{\omega_0^2} \\ &= B + \frac{a_0}{\omega_0^2} \\ \Rightarrow B &= -\frac{a_0}{\omega_0^2} \\ x_h(t) + x_p(t) &= \frac{a_0}{\omega_0^2}(1 - \cos\omega_0 t) \\ x(t) &= \frac{a_0}{\omega_0^2}(1 - \cos\omega_0 t) . \end{aligned}$$

From this solution we deduce that, at $t = \frac{2\pi}{\omega_0}$,

$$\begin{aligned} x\big(t = \tfrac{2\pi}{\omega_0}\big) &= 0 \\ \dot{x}\big(t = \tfrac{2\pi}{\omega_0}\big) &= 0 . \end{aligned}$$

After the push, the equation of motion is

$$\ddot{x} + \omega_0^2 x = 0 .$$

The general solution is

$$x_h = A\cos(\omega_0 t + \alpha) .$$

Applying the initial conditions at $t = \frac{2\pi}{\omega_0}$, *i.e.* the final conditions of the push,

$$\begin{aligned} 0 &= x\big(t = \tfrac{2\pi}{\omega_0}\big) \\ &= A\cos(2\pi + \alpha) \\ &= A\cos\alpha \\ \Rightarrow \alpha &= \tfrac{\pi}{2} \;\text{ or }\; A = 0 \\ 0 &= \dot{x}\big(t = \tfrac{2\pi}{\omega_0}\big) \\ &= -A\omega_0 \sin(2\pi + \alpha) \\ \Rightarrow \alpha &= 0 \;\text{ or }\; A = 0 . \end{aligned}$$

The only mutually consistent way to satisfy both boundary conditions is $A = 0$, so

$$x\big(t > \tfrac{2\pi}{\omega_0}\big) = 0 .$$